THE EFFECT OF TRANSVERSE STRAINS IN THE MECHANICS OF A SOLID MEDIUM

A. F. Revuzhenko and E. I. Shemyakin

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The effect of transverse strains for elastic and plastic bodies is discussed. It is shown that for plastic bodies transverse strains have the same nature as longitudinal strains – rotation of elastic elements cut out by Lüders lines. For elastic bodies longitudinal (active) and transverse (passive) strains have a different nature, and the emergence of internal forces is connected with the gradients of the active but not total displacements. Accordingly the equations of state are written in the form of connections of the stresses with the active strains, and connections of the stresses with the passive strains. It is shown that the connection between stresses and strains calculated with respect to an arbitrary plane has a physical meaning only for certain (defining) planes. The strains calculated on the rest of the planes have no physical meaning; their role is reduced to an invariant description of the processes taking place on the defining planes. Therefore, for the description of the processes taking place on all planes, a single strain tensor is insufficient. It is shown that, in the case of a linearly elastic body, a tensor of the active and a tensor of the passive strains are sufficient.

1. We consider the strain of an elastic solid body. The mathematical theory of elasticity is usually constructed as follows [1]:

1) the concepts of stress and strain tensors are introduced; here the strain tensor by definition characterizes the variation of the distances between parts of closely located points of the elastic body;

2) it is asserted that the stress tensor is a certain function of the strain tensor and the form of this function is postulated.

From a physical viewpoint such a course of setting up a model of an elastic body is either contradictory or it implicitly contains certain additional hypotheses. Indeed, from the last hypothesis and the definition of a strain tensor we can conclude that the only cause of the emergence of internal forces in a body is the variation of the distances between all possible pairs of its closely located points. Hence it follows that from the known strains the forces on any plane can be calculated by two methods: either in terms of the function introduced by the hypothesis 2, or in terms of functions which characterize the interaction of material points of the elastic body.

If, following [2], we assume that only pairs of closely located points interact, then we obtain a model of the elastic body with a single parameter C. If we assume that foursomes of closely located points interact, one of which lies at the vertex of a rectangular reference frame and the three others lie on its sides, then in the isotropic case we obtain a model of the body with two parameters (E and ν). If E and ν are constant, then both methods of determining the stresses lead to the same result. If E and ν vary in the straining process then the internal stresses calculated by the second method do not form a tensor, and the hypotheses 1 and 2 contradict the physical meaning of strain contained by them.

The hypothesis 2 can be altered as follows: we assume that internal forces arise only on certain (defining) planes as a result of the variation of the distances between closely located points of the body; the forces and strains on the remaining planes must be calculated according to the rules of tensor projection in

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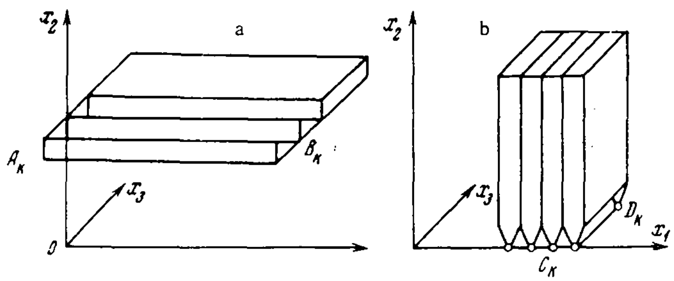

Fig. 1

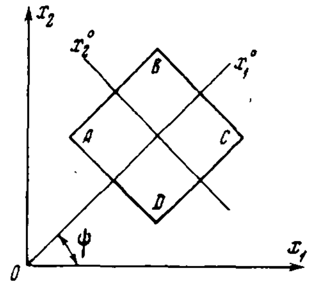

Fig. 2

terms of the forces and strains on the defining planes. Consequently, only on the defining planes the connection between the stresses and strains has a physical meaning; on all other planes this connection has merely a formal meaning. This can be interpreted by the following example: let the kinematics of strain of a solid medium be given by the equations

$$u_1 = \gamma(t)x_2, \quad u_2 \equiv 0, \quad u_1 \equiv 0 \tag{1.1}$$

where u_1, u_2, and u_3 are the components of the displacement vector, and $\gamma(t)$ is a smooth function of the time.

The motion (1.1) can be realized by thin rigid plates A_kB_k between which certain forces (Fig. 1a) act. For such a solid medium only shear on planes parallel to the $0x_1x_3$ plane has a physical meaning. Shear and elongation on other planes have merely a formal meaning. Therefore, the connection between stresses and strains has a physical meaning only for planes parallel to the $0x_1x_3$ plane; these planes in the given case will be defining planes.

In the case of small values of $|\gamma(t)|$ the strain (1.1) can also be realized by rotation of thin rigid plates about the points C_k, D_k (Fig. 1b). In this case planes parallel to the $0x_2x_3$ plane will be defining planes.

The problem of finding defining planes has no single-values solution in the framework of phenomenological representations. For its solution we require information about the physical mechanism of strain of the material under investigation.

We assume that the planes of principal stresses are the defining planes for an elastic material. We consider the plane case of strain. We also assume that before deformation the elastic body is homogeneous and isotropic. We isolate an element ABCD and consider its behavior under the effect of external loads (Fig. 2). If normal forces are applied to the sides AD and BC, then not only the sides AD and BC but also the sides AB and DC are displaced. The displacement of the plane AB is called passive displacement, since the plane AB is free from stresses. The internal forces in the body arise as a result of appearance of gradients of active but not total displacements. In [3], which contains a detailed bibliography, the causes of the emergence of passive displacements are investigated.

In the case of the classical approach it is sufficient to assume that the only cause of a passive displacement is the active displacement of an orthogonal plane. In the general case the displacement of each point of an element ABCD has two components — an active component $\bar{q}$ and a passive component $\bar{p}$. We introduce the concepts of active and passive strains of an element ABCD

$$q_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right), \quad p_{ij} = \frac{1}{2}\left(\frac{\partial p_i}{\partial x_j} + \frac{\partial p_j}{\partial x_i}\right), \quad i, j = 1, 2 \tag{1.2}$$

Here ε_{ij}, p_{ij}, q_{ij}, i, j = 1, 2 are the components of the total, passive and active strains.

In the $0x_1^\circ x_2^\circ$ coordinate system (Fig. 2) the equations

$$\begin{gathered}\varepsilon_{ij}^\circ = p_{ij}^\circ + q_{ij}^\circ, \quad \sigma_{12}^\circ = 0, \; p_{12}^\circ = 0, \quad q_{12}^\circ = 0 \\ q_{11}^\circ = \sigma_{11}^\circ / E_1, \quad q_{22}^\circ = \sigma_{22}^\circ / E_2, \quad p_{11}^\circ = -\nu_{12}q_{22}^\circ, \\ p_{22}^\circ = -\nu_{21}q_{11}^\circ\end{gathered} \tag{1.3}$$

are satisfied.

In the original $0x_1x_2$ coordinate system Eqs. (1.3) are transformed into

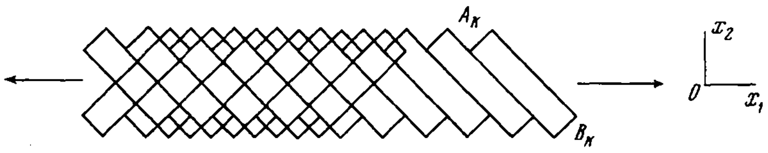

Fig. 3

$$\varepsilon_{11} = p_{11} + q_{11}, \quad \varepsilon_{22} = p_{22} + q_{22}, \quad \varepsilon_{12} = p_{12} + q_{12}$$

$$\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} = \operatorname{tg} 2\psi, \quad \frac{2p_{12}}{p_{11} - p_{22}} = \operatorname{tg} 2\psi, \quad \frac{2q_{12}}{q_{11} - q_{22}} = \operatorname{tg} 2\psi$$

$$\frac{q_{11} + q_{22}}{2} + \frac{q_{11} - q_{22}}{2} \cos 2\psi + q_{12} \sin 2\psi = \frac{1}{E_1}\left[\frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\psi + \sigma_{12} \sin 2\psi\right]$$

$$\frac{q_{11} + q_{22}}{2} - \frac{q_{11} - q_{22}}{2} \cos 2\psi - q_{12} \sin 2\psi = \frac{1}{E_2}\left[\frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\psi - \sigma_{12} \sin 2\psi\right]$$

$$\frac{p_{11} + p_{22}}{2} + \frac{p_{11} - p_{22}}{2} \cos 2\psi + p_{12} \sin 2\psi = -\nu_{12}\left[\frac{q_{11} + q_{22}}{2} - \frac{q_{11} - q_{22}}{2} \cos 2\psi - q_{12} \sin 2\psi\right]$$

$$\frac{p_{11} + p_{22}}{2} - \frac{p_{11} - p_{22}}{2} \cos 2\psi - p_{12} \sin 2\psi = -\nu_{21}\left[\frac{q_{11} + q_{22}}{2} + \frac{q_{11} - q_{22}}{2} \cos 2\psi + q_{12} \sin 2\psi\right] \tag{1.4}$$

The active and passive strains are kinematically indistinguishable; therefore the continuity conditions must be satisfied only for the sum

$$\partial^2\varepsilon_{11} / \partial x_2^2 + \partial^2\varepsilon_{22} / \partial x_1^2 = 2\partial^2\varepsilon_{12} / \partial x_1 \partial x_2 \tag{1.5}$$

The equilibrium equations

$$\partial\sigma_{11} / \partial x_1 + \partial\sigma_{12} / \partial x_2 + \rho X_1 = 0,$$
$$\partial\sigma_{12} / \partial x_1 + \partial\sigma_{22} / \partial x_2 + \rho X_2 = 0 \tag{1.6}$$

where ρX_1, ρX_2 are the components of the body forces and σ_{ij} are stresses, close to the system (1.4) and (1.5).

For physically nonlinear materials the coefficients E_i, ν_{ij} depend on the stresses. If $E_1 \neq E_2$ or $\nu_{12} \neq \nu_{21}$, then an elastic body becomes anisotropic as a result of strain. The problem concerned with the behavior of such a body under a subsequent complex loading requires a special experimental investigation. If $E_1 = E_2 = E$, $\nu_{12} = \nu_{21} = \nu$, then Eq. (1.4) can be simplified.

$$\varepsilon_{11} = p_{11} + q_{11}, \quad \varepsilon_{22} = p_{22} + q_{22}, \quad \varepsilon_{12} = p_{12} + q_{12}$$
$$q_{11} = \sigma_{11} / E, \quad q_{22} = \sigma_{22} / E, \quad q_{12} = \sigma_{12} / E$$
$$p_{11} = -\nu q_{22}, \quad p_{22} = -\nu q_{11}, \quad p_{12} = +\nu q_{12} \tag{1.7}$$

If E = const, ν = const, then we can assume that all planes are defining planes. In this case, for a defining plane, we must introduce the concepts of active and passive shears q_{12}, p_{12}, with

$$q_{12} = \sigma_{12} / E, \qquad p_{12} = \nu q_{12}$$

2. We shall consider the role of the effect of transverse strains for plastic bodies. Experiments show that the mechanism of plastic strain is connected with the motion of dislocations in certain directions [4], i.e., with shear of the material along certain (defining) planes. Planes of maximum shear play the role of defining planes for plastic materials. In Fig. 3 we have schematically depicted the plastic stretching of an incompressible flat strip. The longitudinal and transverse strains reflect the same process – rotation of the elements A_kB_k. When the elements rotate, their projection on the $0x_1$ axis increases, while that on the $0x_2$ axis decreases. The first gives the elongation strain while the other gives the transverse strain. For plastic bodies (including compressible) the longitudinal and transverse strains are equivalent, and a formal separation of the strains into active and passive components would have no physical meaning.

In the case of plane limiting strain of incoherent dry-running materials, the role of defining planes is played by nonorthogonal slip planes. Deformation takes place as a result of change in the volume and slid-

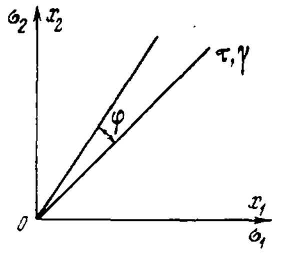

Fig. 4

ing of the material along the defining planes. The longitudinal and transverse strains have a common origin, and separation of the strains into active and passive components is also inadvisable. Analyzing the processes on the defining planes and using the rules of tensor projection, we can set up a model of an incoherent soil.

3. The three-dimensional case of strain of elastic materials is considered analogously to the plane case. In three-dimensional plastic (dry-running) materials we distinguish between cases of incomplete and complete plasticity [5]. In the case of complete plasticity the transverse strains have the same structure as the longitudinal strains. In the case of incomplete plasticity, with respect to the direction of action of the elastic connection and the orthogonal directions, we can separate the active and passive components and connect the corresponding stresses only with the active strains. The strain in a plane that is orthogonal to the elastic direction is analogous to plane plastic strain.

4. The analysis of the effect of transverse strains leads to the following conclusions:

1) for materials whose strain mechanism is connected with the variation of the distances between closely located material points (i.e., with the extension and compression of the fibers), the longitudinal (active) and transverse (passive) strains have a different origin. Internal forces in such materials arise as a result of the emergence of gradients of the active and not the total displacements; strain equations of these materials must be written in a form analogous to Eq. (1.7);

2) for materials whose strain mechanism is connected with shear, the longitudinal and transverse strains have the same origin. The straining process of such materials must be expressed in terms of total strains. The strain tensor (more precisely, its deviator) must be interpreted as an invariant characteristic of shear on different planes;

3) cases are possible where the straining mechanism of the material is different in different directions. Accordingly, the transverse strain is made up of parts having a different origin;

4) if the straining process of the body is expressed in terms of a certain function (functional) of the stress and strain tensors, then this function (functional) has a physical meaning only for certain (defining) planes and directions in them. The connection between the stresses and strains on the remaining planes has no independent physical meaning; the stresses and strains on them serve for invariant description of the processes taking place on the defining planes. The strain equations must be written in the form analogous to Eq. (1.3).

The use of stress and strain tensors allows us to describe in invariant form the processes taking place only on the defining planes. Processes taking place on the rest of the planes remain unknown in the case of such description. (This does not apply to the components of the stress tensor themselves.) The validity of the conclusion 4 for elastic bodies is shown in Section 1; for plastic bodies the conclusion 4 follows from an example.

We consider plane strain of a plastic material under proportional loading (Fig. 4). Let τ and γ be the maximum shear stress and shear strain. Then a decrease in the modulus $\mu = \partial\tau/\partial\gamma$ signifies weakening of the material along the maximum shear plane. On the plane $\varphi \neq 0$ the shear stress and shear strain are $\tau \cos 2\varphi$ and $\gamma \cos 2\varphi$. We now assume that the stress and strain tensors describe processes taking place on all planes. Consequently, weakening of the plane φ is characterized by the shear modulus.

$$\mu_\varphi = \partial(\varepsilon \cos 2\varphi) / \partial(\gamma \cos 2\varphi) = \mu \qquad (4.1)$$

From Eq. (4.1) it follows that the material remains isotropic. Experiments [4] show the converse. Consequently, the assumption is incorrect and the conclusion 4 is valid also for the plastic materials. We note that the problem of arbitrary loading of solid media reduces to an adequate description of the processes taking place on all planes, i.e., to the determination of the denominator in Eq. (4.1). It is not possible to do this by means of a single strain tensor. In the case of a linearly elastic body, the tensor of active strains turned out to be sufficient for the solution of this problem. For more general cases the question remains open.

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